A LEVEL MATHEMATICS TRANSITION BOOKLET

"MATHEMATICIANS AREN'T PEOPLE WHO FIND MATHS EASY. THEY'RE PEOPLE WHO ENJOY HOW HARD IT IS."

- To prepare yourselves for the rigour of the A Level course you will need to complete this Transition Booklet
- Mark this yourself with red pen and highlight any areas of difficulty below
- This will be checked in the first week back
- Your understanding of these key concepts and skills will be assessed using our **Mathematics Induction Test**, due to take place within the first fortnight in September.

Skills Check	RAG Rating
Surds	
Indices	
Completing the square	
Sketching quadratics	
Linear/Quadratic simultaneous equations	
Quadratic inequalities	
Rearranging equations	
Algebraic fractions	
Straight line graphs	

"ANYONE WHO HAS NEVER MADE A MISTAKE HAS NEVER TRIED ANYTHING NEW."

Albert Einstein

Surds and rationalising the denominator

A LEVEL LINKS
Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5}$, etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\frac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd $\sqrt{b}$
- To rationalise $\frac{a}{b+\sqrt{c}}$ you multiply the numerator and denominator by $b-\sqrt{c}$

Examples

Example 1 Simplify $\sqrt{50}$

$\sqrt{50} = \sqrt{25 \times 2}$	**1** Choose two numbers that are factors of 50. One of the factors must be a square number
$= \sqrt{25} \times \sqrt{2}$	**2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 5 \times \sqrt{2}$	**3** Use $\sqrt{25} = 5$
$= 5\sqrt{2}$	

Example 2 Simplify $\sqrt{147} - 2\sqrt{12}$

$\sqrt{147} - 2\sqrt{12}$ $= \sqrt{49 \times 3} - 2\sqrt{4 \times 3}$	**1** Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number
$= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3}$	**2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3}$	**3** Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$
$= 7\sqrt{3} - 4\sqrt{3}$ $= 3\sqrt{3}$	**4** Collect like terms

Example 3 Simplify $\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$

$\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$ $=\sqrt{49}-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}-\sqrt{4}$ $=7-2$ $=5$	**1** Expand the brackets. A common mistake here is to write $\left(\sqrt{7}\right)^2=49$ **2** Collect like terms: $-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}$ $=-\sqrt{7}\sqrt{2}+\sqrt{7}\sqrt{2}=0$

Example 4 Rationalise $\dfrac{1}{\sqrt{3}}$

$\dfrac{1}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$ $=\dfrac{1\times\sqrt{3}}{\sqrt{9}}$ $=\dfrac{\sqrt{3}}{3}$	**1** Multiply the numerator and denominator by $\sqrt{3}$ **2** Use $\sqrt{9}=3$

Example 5 Rationalise and simplify $\dfrac{\sqrt{2}}{\sqrt{12}}$

$\dfrac{\sqrt{2}}{\sqrt{12}}=\dfrac{\sqrt{2}}{\sqrt{12}}\times\dfrac{\sqrt{12}}{\sqrt{12}}$ $=\dfrac{\sqrt{2}\times\sqrt{4\times3}}{12}$ $=\dfrac{2\sqrt{2}\sqrt{3}}{12}$ $=\dfrac{\sqrt{2}\sqrt{3}}{6}$	**1** Multiply the numerator and denominator by $\sqrt{12}$ **2** Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number **3** Use the rule $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$ **4** Use $\sqrt{4}=2$ **5** Simplify the fraction: $\dfrac{2}{12}$ simplifies to $\dfrac{1}{6}$

Example 6 Rationalise and simplify $\dfrac{3}{2+\sqrt{5}}$

$\dfrac{3}{2+\sqrt{5}} = \dfrac{3}{2+\sqrt{5}} \times \dfrac{2-\sqrt{5}}{2-\sqrt{5}}$	**1** Multiply the numerator and denominator by $2-\sqrt{5}$
$= \dfrac{3(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}$	**2** Expand the brackets
$= \dfrac{6-3\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-5}$	**3** Simplify the fraction
$= \dfrac{6-3\sqrt{5}}{-1}$ $= 3\sqrt{5}-6$	**4** Divide the numerator by -1 Remember to change the sign of all terms when dividing by -1

Practice

1 Simplify.

a $\sqrt{45}$ **b** $\sqrt{125}$

c $\sqrt{48}$ **d** $\sqrt{175}$

e $\sqrt{300}$ **f** $\sqrt{28}$

g $\sqrt{72}$ **h** $\sqrt{162}$

> **Hint**
>
> One of the two numbers you choose at the start must be a square number.

2 Simplify.

a $\sqrt{72}+\sqrt{162}$ **b** $\sqrt{45}-2\sqrt{5}$

c $\sqrt{50}-\sqrt{8}$ **d** $\sqrt{75}-\sqrt{48}$

e $2\sqrt{28}+\sqrt{28}$ **f** $2\sqrt{12}-\sqrt{12}+\sqrt{27}$

> **Watch out!**
>
> Check you have chosen the highest square number at the start.

3 Expand and simplify.

a $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **b** $(3+\sqrt{3})(5-\sqrt{12})$

c $(4-\sqrt{5})(\sqrt{45}+2)$ **d** $(5+\sqrt{2})(6-\sqrt{8})$

4 Rationalise and simplify, if possible.

a $\frac{1}{\sqrt{5}}$ b $\frac{1}{\sqrt{11}}$

c $\frac{2}{\sqrt{7}}$ d $\frac{2}{\sqrt{8}}$

e $\frac{2}{\sqrt{2}}$ f $\frac{5}{\sqrt{5}}$

g $\frac{\sqrt{8}}{\sqrt{24}}$ h $\frac{\sqrt{5}}{\sqrt{45}}$

5 Rationalise and simplify.

a $\frac{1}{3-\sqrt{5}}$ b $\frac{2}{4+\sqrt{3}}$ c $\frac{6}{5-\sqrt{2}}$

Extend

6 Expand and simplify $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)$

7 Rationalise and simplify, if possible.

a $\frac{1}{\sqrt{9}-\sqrt{8}}$ b $\frac{1}{\sqrt{x}-\sqrt{y}}$

Answers

1 **a** $3\sqrt{5}$ **b** $5\sqrt{5}$
c $4\sqrt{3}$ **d** $5\sqrt{7}$
e $10\sqrt{3}$ **f** $2\sqrt{7}$
g $6\sqrt{2}$ **h** $9\sqrt{2}$

2 **a** $15\sqrt{2}$ **b** $\sqrt{5}$
c $3\sqrt{2}$ **d** $\sqrt{3}$
e $6\sqrt{7}$ **f** $5\sqrt{3}$

3 **a** -1 **b** $9-\sqrt{3}$
c $10\sqrt{5}-7$ **d** $26-4\sqrt{2}$

4 **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{11}}{11}$
c $\frac{2\sqrt{7}}{7}$ **d** $\frac{\sqrt{2}}{2}$
e $\sqrt{2}$ **f** $\sqrt{5}$
g $\frac{\sqrt{3}}{3}$ **h** $\frac{1}{3}$

5 **a** $\frac{3+\sqrt{5}}{4}$ **b** $\frac{2(4-\sqrt{3})}{13}$ **c** $\frac{6(5+\sqrt{2})}{23}$

6 $x-y$

7 **a** $3+2\sqrt{2}$ **b** $\frac{\sqrt{x}+\sqrt{y}}{x-y}$

Rules of indices

A LEVEL LINKS
Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the nth root of a
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \dfrac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

Examples

Example 1 Evaluate 10^0

$10^0 = 1$	Any value raised to the power of zero is equal to 1

Example 2 Evaluate $9^{\frac{1}{2}}$

$9^{\frac{1}{2}} = \sqrt{9}$ $= 3$	Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$

Example 3 Evaluate $27^{\frac{2}{3}}$

$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ $= 3^2$ $= 9$	**1** Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ **2** Use $\sqrt[3]{27} = 3$

Example 4 Evaluate 4^{-2}

$4^{-2}=\frac{1}{4^2}$ $=\frac{1}{16}$	**1** Use the rule $a^{-m}=\frac{1}{a^m}$ **2** Use $4^2=16$

Example 5 Simplify $\frac{6x^5}{2x^2}$

$\frac{6x^5}{2x^2}=3x^3$	$6 \div 2 = 3$ and use the rule $\frac{a^m}{a^n}=a^{m-n}$ to give $\frac{x^5}{x^2}=x^{5-2}=x^3$

Example 6 Simplify $\frac{x^3\times x^5}{x^4}$

$\frac{x^3\times x^5}{x^4}=\frac{x^{3+5}}{x^4}=\frac{x^8}{x^4}$ $=x^{8-4}=x^4$	**1** Use the rule $a^m\times a^n=a^{m+n}$ **2** Use the rule $\frac{a^m}{a^n}=a^{m-n}$

Example 7 Write $\frac{1}{3x}$ as a single power of x

$\frac{1}{3x}=\frac{1}{3}x^{-1}$	Use the rule $\frac{1}{a^m}=a^{-m}$, note that the fraction $\frac{1}{3}$ remains unchanged

Example 8 Write $\frac{4}{\sqrt{x}}$ as a single power of x

$\frac{4}{\sqrt{x}}=\frac{4}{x^{\frac{1}{2}}}$ $=4x^{-\frac{1}{2}}$	**1** Use the rule $a^{\frac{1}{n}}=\sqrt[n]{a}$ **2** Use the rule $\frac{1}{a^m}=a^{-m}$

Practice

1 Evaluate.

a 14^0 **b** 3^0 **c** 5^0 **d** x^0

2 Evaluate.

a $49^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $125^{\frac{1}{3}}$ **d** $16^{\frac{1}{4}}$

3 Evaluate.

a $25^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $16^{\frac{3}{4}}$

4 Evaluate.

a 5^{-2} **b** 4^{-3} **c** 2^{-5} **d** 6^{-2}

5 Simplify.

a $\dfrac{3x^2 \times x^3}{2x^2}$ **b** $\dfrac{10x^5}{2x^2 \times x}$

c $\dfrac{3x \times 2x^3}{2x^3}$ **d** $\dfrac{7x^3y^2}{14x^5y}$

e $\dfrac{y^2}{y^{\frac{1}{2}} \times y}$ **f** $\dfrac{c^{\frac{1}{2}}}{c^2 \times c^{\frac{3}{2}}}$

g $\dfrac{\left(2x^2\right)^3}{4x^0}$ **h** $\dfrac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^{-2} \times x^3}$

Watch out!

Remember that any value raised to the power of zero is 1. This is the rule $a^0 = 1$.

6 Evaluate.

a $4^{-\frac{1}{2}}$ **b** $27^{-\frac{2}{3}}$ **c** $9^{-\frac{1}{2}} \times 2^3$

d $16^{\frac{1}{4}} \times 2^{-3}$ **e** $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}$ **f** $\left(\dfrac{27}{64}\right)^{-\frac{2}{3}}$

7 Write the following as a single power of x.

a $\dfrac{1}{x}$ **b** $\dfrac{1}{x^7}$ **c** $\sqrt[4]{x}$

d $\sqrt[5]{x^2}$ **e** $\dfrac{1}{\sqrt[3]{x}}$ **f** $\dfrac{1}{\sqrt[3]{x^2}}$

8 Write the following without negative or fractional powers.

a x^{-3} **b** x^0 **c** $x^{\frac{1}{5}}$

d $x^{\frac{2}{5}}$ **e** $x^{-\frac{1}{2}}$ **f** $x^{-\frac{3}{4}}$

9 Write the following in the form ax^n.

a $5\sqrt{x}$ **b** $\frac{2}{x^3}$ **c** $\frac{1}{3x^4}$

d $\frac{2}{\sqrt{x}}$ **e** $\frac{4}{\sqrt[3]{x}}$ **f** 3

Extend

10 Write as sums of powers of x.

a $\frac{x^5+1}{x^2}$ **b** $x^2\left(x+\frac{1}{x}\right)$ **c** $x^{-4}\left(x^2+\frac{1}{x^3}\right)$

Answers

1	a	1	b	1	c	1	d	1
2	a	7	b	4	c	5	d	2
3	a	125	b	32	c	343	d	8
4	a	$\frac{1}{25}$	b	$\frac{1}{64}$	c	$\frac{1}{32}$	d	$\frac{1}{36}$
5	a	$\frac{3x^3}{2}$	b	$5x^2$				
	c	$3x$	d	$\frac{y}{2x^2}$				
	e	$y^{\frac{1}{2}}$	f	c^{-3}				
	g	$2x^6$	h	x				
6	a	$\frac{1}{2}$	b	$\frac{1}{9}$	c	$\frac{8}{3}$		
	d	$\frac{1}{4}$	e	$\frac{4}{3}$	f	$\frac{16}{9}$		
7	a	x^{-1}	b	x^{-7}	c	$x^{\frac{1}{4}}$		
	d	$x^{\frac{2}{5}}$	e	$x^{-\frac{1}{3}}$	f	$x^{-\frac{2}{3}}$		
8	a	$\frac{1}{x^3}$	b	1	c	$\sqrt[5]{x}$		
	d	$\sqrt[5]{x^2}$	e	$\frac{1}{\sqrt{x}}$	f	$\frac{1}{\sqrt[4]{x^3}}$		
9	a	$5x^{\frac{1}{2}}$	b	$2x^{-3}$	c	$\frac{1}{3}x^{-4}$		
	d	$2x^{-\frac{1}{2}}$	e	$4x^{-\frac{1}{3}}$	f	$3x^0$		
10	a	$x^3 + x^{-2}$	b	$x^3 + x$	c	$x^{-2} + x^{-7}$		

Completing the square

A LEVEL LINKS
Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$ $= (x + 3)^2 - 9 - 2$ $= (x + 3)^2 - 11$	**1** Write $x^2 + bx + c$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ **2** Simplify

Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$	**1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$
$= 2\left(x^2 - \frac{5}{2}x\right) + 1$	**2** Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
$= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$	
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$	**3** Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	**4** Simplify

Practice

1 Write the following quadratic expressions in the form $(x + p)^2 + q$

a $x^2 + 4x + 3$

b $x^2 - 10x - 3$

c $x^2 - 8x$

d $x^2 + 6x$

e $x^2 - 2x + 7$

f $x^2 + 3x - 2$

2 Write the following quadratic expressions in the form $p(x + q)^2 + r$

a $2x^2 - 8x - 16$

b $4x^2 - 8x - 16$

c $3x^2 + 12x - 9$

d $2x^2 + 6x - 8$

3 Complete the square.

a $2x^2 + 3x + 6$

b $3x^2 - 2x$

c $5x^2 + 3x$

d $3x^2 + 5x + 3$

Extend

4 Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

Answers

1 **a** $(x+2)^2 - 1$ **b** $(x-5)^2 - 28$

c $(x-4)^2 - 16$ **d** $(x+3)^2 - 9$

e $(x-1)^2 + 6$ **f** $\left(x+\frac{3}{2}\right)^2 - \frac{17}{4}$

2 **a** $2(x-2)^2 - 24$ **b** $4(x-1)^2 - 20$

c $3(x+2)^2 - 21$ **d** $2\left(x+\frac{3}{2}\right)^2 - \frac{25}{2}$

3 **a** $2\left(x+\frac{3}{4}\right)^2 + \frac{39}{8}$ **b** $3\left(x-\frac{1}{3}\right)^2 - \frac{1}{3}$

c $5\left(x+\frac{3}{10}\right)^2 - \frac{9}{20}$ **d** $3\left(x+\frac{5}{6}\right)^2 + \frac{11}{12}$

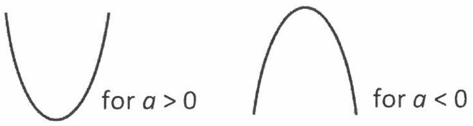
4 $(5x+3)^2 + 3$

Sketching quadratic graphs

A LEVEL LINKS
Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

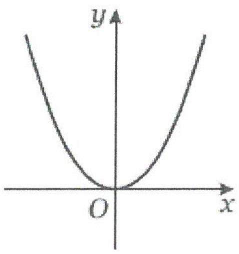
- The graph of the quadratic function $y = ax^2 + bx + c$, where $a \neq 0$, is a curve called a parabola.
- Parabolas have a line of symmetry and a shape as shown.



- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- To find the coordinates of the maximum or minimum point (turning points) of a quadratic curve (parabola) you can use the completed square form of the function.

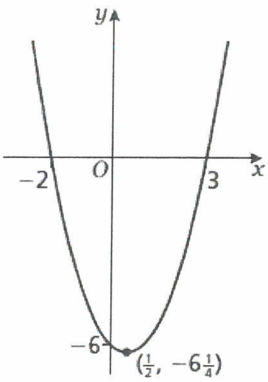
Examples

Example 1 Sketch the graph of $y = x^2$.

	The graph of $y = x^2$ is a parabola. When $x = 0$, $y = 0$. $a = 1$ which is greater than zero, so the graph has the shape: 

Example 2 Sketch the graph of $y = x^2 - x - 6$.

When $x = 0$, $y = 0^2 - 0 - 6 = -6$ So the graph intersects the y-axis at $(0, -6)$ When $y = 0$, $x^2 - x - 6 = 0$	**1** Find where the graph intersects the y-axis by substituting $x = 0$. **2** Find where the graph intersects the x-axis by substituting $y = 0$.
$(x + 2)(x - 3) = 0$	**3** Solve the equation by factorising.
$x = -2$ or $x = 3$	**4** Solve $(x + 2) = 0$ and $(x - 3) = 0$.
So, the graph intersects the x-axis at $(-2, 0)$ and $(3, 0)$	**5** $a = 1$ which is greater than zero, so the graph has the shape:  *(continued on next page)*
$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 6$	**6** To find the turning point, complete the square.

$= \left(x-\frac{1}{2}\right)^2 - \frac{25}{4}$ When $\left(x-\frac{1}{2}\right)^2 = 0$, $x = \frac{1}{2}$ and $y = -\frac{25}{4}$, so the turning point is at the point $\left(\frac{1}{2}, -\frac{25}{4}\right)$ 	**7** The turning point is the minimum value for this expression and occurs when the term in the bracket is equal to zero.

Practice

1 Sketch the graph of $y = -x^2$.

2 Sketch each graph, labelling where the curve crosses the axes.

a $y = (x + 2)(x - 1)$ **b** $y = x(x - 3)$ **c** $y = (x + 1)(x + 5)$

3 Sketch each graph, labelling where the curve crosses the axes.

a $y = x^2 - x - 6$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 - 4$

d $y = x^2 + 4x$ **e** $y = 9 - x^2$ **f** $y = x^2 + 2x - 3$

4 Sketch the graph of $y = 2x^2 + 5x - 3$, labelling where the curve crosses the axes.

Extend

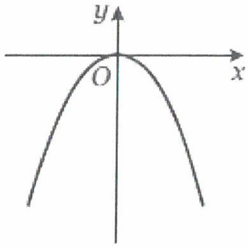
5 Sketch each graph. Label where the curve crosses the axes and write down the coordinates of the turning point.

a $y = x^2 - 5x + 6$ **b** $y = -x^2 + 7x - 12$ **c** $y = -x^2 + 4x$

6 Sketch the graph of $y = x^2 + 2x + 1$. Label where the curve crosses the axes and write down the equation of the line of symmetry.

Answers

1



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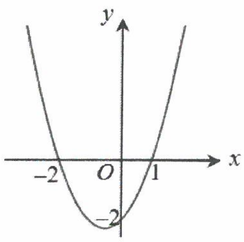

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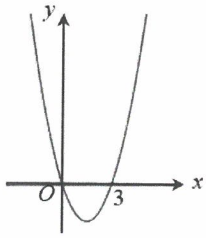

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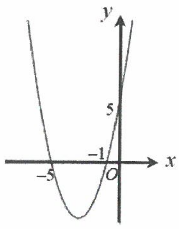

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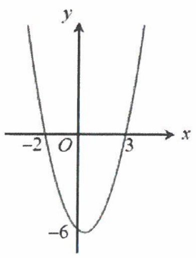

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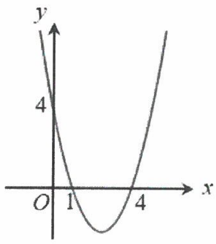

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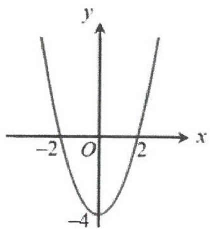

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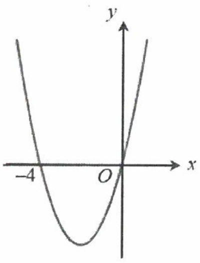

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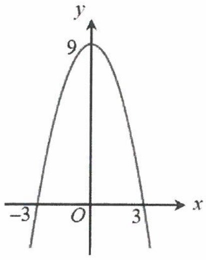

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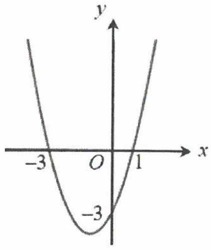

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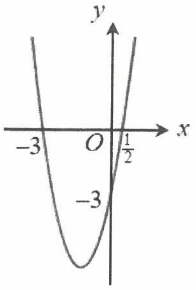
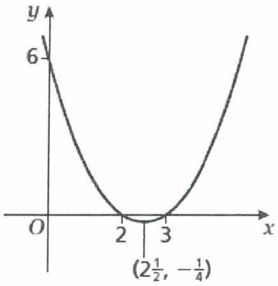


5 **a**



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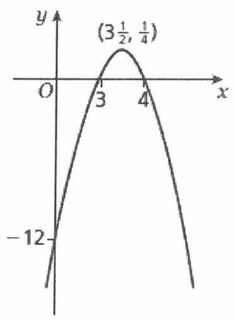

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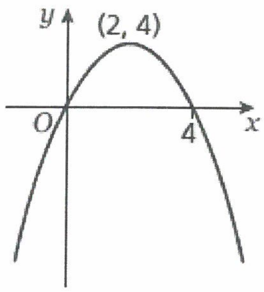

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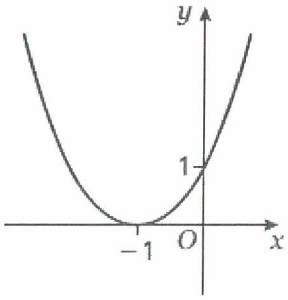


Line of symmetry at $x = -1$.

Solving linear and quadratic simultaneous equations

A LEVEL LINKS
Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1 Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$	**1** Substitute $x + 1$ for y into the second equation.
$x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$	**2** Expand the brackets and simplify.
$2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$	**3** Factorise the quadratic equation.
So $x = 2$ or $x = -3$	**4** Work out the values of x.
Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$	**5** To find the value of y, substitute both values of x into one of the original equations.
So the solutions are $x = 2,\ y = 3$ and $x = -3, y = -2$	
Check: equation 1: $3 = 2 + 1$ YES and $-2 = -3 + 1$ YES equation 2: $2^2 + 3^2 = 13$ YES and $(-3)^2 + (-2)^2 = 13$ YES	**6** Substitute both pairs of values of x and y into both equations to check your answers.

Example 2 Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \dfrac{5-3y}{2}$	**1** Rearrange the first equation.
$2y^2 + \left(\dfrac{5-3y}{2}\right)y = 12$	**2** Substitute $\dfrac{5-3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y.
$2y^2 + \dfrac{5y-3y^2}{2} = 12$ $4y^2 + 5y - 3y^2 = 24$ $y^2 + 5y - 24 = 0$	**3** Expand the brackets and simplify.
$(y + 8)(y - 3) = 0$	**4** Factorise the quadratic equation.
So $y = -8$ or $y = 3$	**5** Work out the values of y.
Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$ So the solutions are $x = 14.5$, $y = -8$ and $x = -2$, $y = 3$	**6** To find the value of x, substitute both values of y into one of the original equations.
Check: equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES and $2 \times (-2) + 3 \times 3 = 5$ YES equation 2: $2\times(-8)^2 + 14.5\times(-8) = 12$ YES and $2 \times (3)^2 + (-2) \times 3 = 12$ YES	**7** Substitute both pairs of values of x and y into both equations to check your answers.

Practice

Solve these simultaneously equations.

1 $y = 2x + 1$
$x^2 + y^2 = 10$

2 $y = 6 - x$
$x^2 + y^2 = 20$

3 $y = x - 3$
$x^2 + y^2 = 5$

4 $y = 9 - 2x$
$x^2 + y^2 = 17$

5 $y = 3x - 5$
$y = x^2 - 2x + 1$

6 $y = x - 5$
$y = x^2 - 5x - 12$

7 $y = x + 5$
$x^2 + y^2 = 25$

8 $y = 2x - 1$
$x^2 + xy = 24$

9 $y = 2x$
$y^2 - xy = 8$

10 $2x + y = 11$
$xy = 15$

Extend

11 $x - y = 1$
$x^2 + y^2 = 3$

12 $y - x = 2$
$x^2 + xy = 3$

Answers

1 $x = 1, y = 3$

$x = -\frac{9}{5}, y = -\frac{13}{5}$

2 $x = 2, y = 4$

$x = 4, y = 2$

3 $x = 1, y = -2$

$x = 2, y = -1$

4 $x = 4, y = 1$

$x = \frac{16}{5}, y = \frac{13}{5}$

5 $x = 3, y = 4$

$x = 2, y = 1$

6 $x = 7, y = 2$

$x = -1, y = -6$

7 $x = 0, y = 5$

$x = -5, y = 0$

8 $x = -\frac{8}{3}, y = -\frac{19}{3}$

$x = 3, y = 5$

9 $x = -2, y = -4$

$x = 2, y = 4$

10 $x = \frac{5}{2}, y = 6$

$x = 3, y = 5$

11 $x = \frac{1+\sqrt{5}}{2}, y = \frac{-1+\sqrt{5}}{2}$

$x = \frac{1-\sqrt{5}}{2}, y = \frac{-1-\sqrt{5}}{2}$

12 $x = \frac{-1+\sqrt{7}}{2}, y = \frac{3+\sqrt{7}}{2}$

$x = \frac{-1-\sqrt{7}}{2}, y = \frac{3-\sqrt{7}}{2}$

Quadratic inequalities

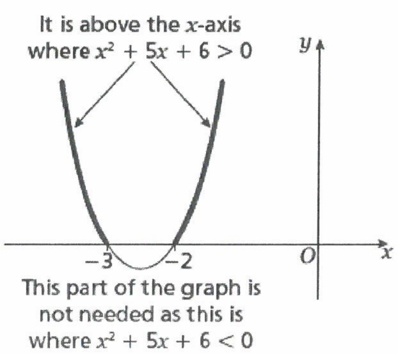
A LEVEL LINKS
Scheme of work: 1d. Inequalities – linear and quadratic (including graphical solutions)

Key points

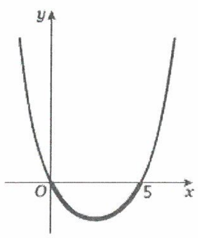
- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

Examples

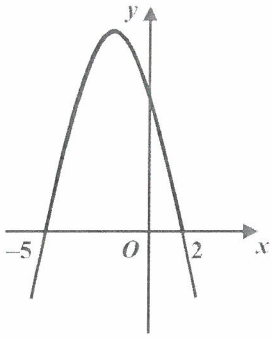
Example 1 Find the set of values of x which satisfy $x^2 + 5x + 6 > 0$

$x^2 + 5x + 6 = 0$ $(x + 3)(x + 2) = 0$ $x = -3$ or $x = -2$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = (x + 3)(x + 2)$ **3** Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$
$x < -3$ or $x > -2$	**4** Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$

Example 2 Find the set of values of x which satisfy $x^2 - 5x \leq 0$

$x^2 - 5x = 0$ $x(x - 5) = 0$ $x = 0$ or $x = 5$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = x(x - 5)$ **3** Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$
$0 \leq x \leq 5$	**4** Write down the values which satisfy the inequality $x^2 - 5x \leq 0$

Example 3 Find the set of values of x which satisfy $-x^2 - 3x + 10 \geq 0$

$-x^2 - 3x + 10 = 0$ $(-x + 2)(x + 5) = 0$ $x = 2$ or $x = -5$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = (-x + 2)(x + 5) = 0$ **3** Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$
$-5 \leq x \leq 2$	**3** Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$

Practice

1 Find the set of values of x for which $(x + 7)(x - 4) \leq 0$

2 Find the set of values of x for which $x^2 - 4x - 12 \geq 0$

3 Find the set of values of x for which $2x^2 - 7x + 3 < 0$

4 Find the set of values of x for which $4x^2 + 4x - 3 > 0$

5 Find the set of values of x for which $12 + x - x^2 \geq 0$

Extend

Find the set of values which satisfy the following inequalities.

6 $x^2 + x \leq 6$

7 $x(2x - 9) < -10$

8 $6x^2 \geq 15 + x$

Answers

1 $-7 \le x \le 4$

2 $x \le -2$ or $x \ge 6$

3 $\frac{1}{2} < x < 3$

4 $x < -\frac{3}{2}$ or $x > \frac{1}{2}$

5 $-3 \le x \le 4$

6 $-3 \le x \le 2$

7 $2 < x < 2\frac{1}{2}$

8 $x \le -\frac{3}{2}$ or $x \ge \frac{5}{3}$

Rearranging equations

A LEVEL LINKS
Scheme of work: 6a. Definition, differentiating polynomials, second derivatives
Textbook: Pure Year 1, 12.1 Gradients of curves

Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

Examples

Example 1 Make t the subject of the formula $v = u + at$.

$v = u + at$ $v - u = at$ $t = \dfrac{v-u}{a}$	**1** Get the terms containing t on one side and everything else on the other side. **2** Divide throughout by a.

Example 2 Make t the subject of the formula $r = 2t - \pi t$.

$r = 2t - \pi t$ $r = t(2 - \pi)$ $t = \dfrac{r}{2-\pi}$	**1** All the terms containing t are already on one side and everything else is on the other side. **2** Factorise as t is a common factor. **3** Divide throughout by $2 - \pi$.

Example 3 Make t the subject of the formula $\dfrac{t+r}{5} = \dfrac{3t}{2}$.

$\dfrac{t+r}{5} = \dfrac{3t}{2}$ $2t + 2r = 15t$ $2r = 13t$ $t = \dfrac{2r}{13}$	**1** Remove the fractions first by multiplying throughout by 10. **2** Get the terms containing t on one side and everything else on the other side and simplify. **3** Divide throughout by 13.

Example 4 Make t the subject of the formula $r = \dfrac{3t+5}{t-1}$.

$r = \dfrac{3t+5}{t-1}$ $r(t-1) = 3t+5$ $rt - r = 3t + 5$ $rt - 3t = 5 + r$ $t(r-3) = 5 + r$ $t = \dfrac{5+r}{r-3}$	**1** Remove the fraction first by multiplying throughout by $t - 1$. **2** Expand the brackets. **3** Get the terms containing t on one side and everything else on the other side. **4** Factorise the LHS as t is a common factor. **5** Divide throughout by $r - 3$.

Practice

Change the subject of each formula to the letter given in the brackets.

1 $C = \pi d$ [d]

2 $P = 2l + 2w$ [w]

3 $D = \dfrac{S}{T}$ [T]

4 $p = \dfrac{q-r}{t}$ [t]

5 $u = at - \dfrac{1}{2}t$ [t]

6 $V = ax + 4x$ [x]

7 $\dfrac{y-7x}{2} = \dfrac{7-2y}{3}$ [y]

8 $x = \dfrac{2a-1}{3-a}$ [a]

9 $x = \dfrac{b-c}{d}$ [d]

10 $h = \dfrac{7g-9}{2+g}$ [g]

11 $e(9+x) = 2e + 1$ [e]

12 $y = \dfrac{2x+3}{4-x}$ [x]

13 Make r the subject of the following formulae.

a $A = \pi r^2$ **b** $V = \dfrac{4}{3}\pi r^3$ **c** $P = \pi r + 2r$ **d** $V = \dfrac{2}{3}\pi r^2 h$

14 Make x the subject of the following formulae.

a $\dfrac{xy}{z} = \dfrac{ab}{cd}$ **b** $\dfrac{4\pi cx}{d} = \dfrac{3z}{py^2}$

15 Make sin B the subject of the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

16 Make cos B the subject of the formula $b^2 = a^2 + c^2 - 2ac\cos B$.

Extend

17 Make x the subject of the following equations.

a $\dfrac{p}{q}(sx+t) = x - 1$ **b** $\dfrac{p}{q}(ax+2y) = \dfrac{3p}{q^2}(x-y)$

Answers

1 $d = \dfrac{C}{\pi}$ **2** $w = \dfrac{P-2l}{2}$ **3** $T = \dfrac{S}{D}$

4 $t = \dfrac{q-r}{p}$ **5** $t = \dfrac{2u}{2a-1}$ **6** $x = \dfrac{V}{a+4}$

7 $y = 2 + 3x$ **8** $a = \dfrac{3x+1}{x+2}$ **9** $d = \dfrac{b-c}{x}$

10 $g = \dfrac{2h+9}{7-h}$ **11** $e = \dfrac{1}{x+7}$ **12** $x = \dfrac{4y-3}{2+y}$

13 **a** $r = \sqrt{\dfrac{A}{\pi}}$ **b** $r = \sqrt[3]{\dfrac{3V}{4\pi}}$

c $r = \dfrac{P}{\pi+2}$ **d** $r = \sqrt{\dfrac{3V}{2\pi h}}$

14 **a** $x = \dfrac{abz}{cdy}$ **b** $x = \dfrac{3dz}{4\pi cpy^2}$

15 $\sin B = \dfrac{b \sin A}{a}$

16 $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

17 **a** $x = \dfrac{q+pt}{q-ps}$ **b** $x = \dfrac{3py + 2pqy}{3p - apq} = \dfrac{y(3+2q)}{3-aq}$

Algebraic Fractions

Many people have only a hazy idea of fractions. That needs improving if you want to go a long way with maths – you will need to be confident in handling fractions consisting of letters as well as numbers.

Remember, first, how to multiply a fraction by an integer. You multiply only the top *[what happens if you multiply both the top and the bottom of a fraction by the same thing?]*

Example 1 Multiply $\frac{4}{29}$ by 3.

Solution $4 \times 3 = 12$, so the answer is $\frac{12}{29}$.

Sometimes you can simplify the answer. If there is a common factor between the denominator (bottom) of the fraction and the number you are multiplying by, you can *divide* by that common factor.

Example 2 Multiply $\frac{7}{39}$ by 3.

Solution $39 \div 3 = 13$, so the answer is $\frac{7}{13}$.

You will remember that when you divide one fraction by another, you turn the one you are dividing by upside down, and multiply. If you are dividing by a whole number, you may need to write it as a fraction.

Example 3 Divide $\frac{7}{8}$ by 5.

Solution $\frac{7}{8} \div 5 = \frac{7}{8} \times \frac{1}{5}$, so the answer is $\frac{7}{40}$.

But if you can, you divide the top of the fraction only.

Example 4 Divide $\frac{20}{43}$ by 5.

Solution $\frac{20}{1} \times \frac{1}{5} = \frac{4}{1}$, so the answer is $\frac{4}{43}$. *Note* that you divide 20 by 5.

Do ***not*** multiply out 5×43; you'll only have to divide it again at the end!

Example 5 Multiply $\dfrac{3x}{7y}$ by 2.

Solution $3 \times 2 = 6x$, so the answer is $\dfrac{6x}{7y}$. *(Not* $\dfrac{6x}{14y}$!)

Example 6 Divide $\dfrac{3y^2}{4x}$ by y.

Solution $\dfrac{3y^2}{4x} \div y = \dfrac{3y^2}{4x} \times \dfrac{1}{y} = \dfrac{3y^2}{4xy} = \dfrac{3y}{4x}$, so the answer is $\dfrac{3y}{4x}$. [Don't forget to simplify.]

Example 7 Divide $\dfrac{PQR}{100}$ by T.

Solution $\dfrac{PQR}{100} \div T = \dfrac{PQR}{100} \times \dfrac{1}{T} = \dfrac{PQR}{100T}$.

Here it would be wrong to say just $\dfrac{\frac{PQR}{100}}{T}$, which is a mix (as well as a mess!)

Double fractions, or mixtures of fractions and decimals, are always wrong.

For instance, if you want to divide $\dfrac{xy}{z}$ by 2, you should not say $\dfrac{0.5xy}{z}$ but $\dfrac{xy}{2z}$.

This sort of thing is extremely important when it comes to rearranging formulae.

Example 8 Make r the subject of the equation $V = \frac{1}{2}\pi r^2 h$.

Solution Multiply by 2: $2V = \pi r^2 h$

Don't "divide by $\frac{1}{2}$

Divide by π and h: $\dfrac{2V}{\pi h} = r^2$

Square root both sides: $r = \sqrt{\dfrac{2V}{\pi h}}$.

You should *not* write the answer as $\sqrt{\dfrac{V}{\frac{1}{2}\pi h}}$ or $\sqrt{\dfrac{2V}{\pi} \div h}$, as these are fractions of fractions.

Make sure, too, that you write the answer properly. If you write $\sqrt{}2V/\pi h$ it's not at all clear that the whole expression has to be square-rooted and you will lose marks.

If you do get a compound fraction (a fraction in which either the numerator or the denominator, or both, contain one or more fractions), you can always simplify it by multiplying all the terms, on both top and bottom, by any *inner denominators*.

Example 9 Simplify $\dfrac{\dfrac{1}{x-1}+1}{\dfrac{1}{x-1}-1}$.

Solution Multiply all four terms, on both top and bottom, by $(x-1)$:

$$\frac{\dfrac{1}{x-1}+1}{\dfrac{1}{x-1}-1} = \frac{\dfrac{(x-1)}{x-1}+(x-1)}{\dfrac{(x-1)}{x-1}-(x-1)}$$

$$= \frac{1+(x-1)}{1-(x-1)}$$

$$= \frac{x}{2-x}$$

You will often want to combine two algebraic expressions, one of which is an algebraic fraction, into a single expression. You will no doubt remember how to add or subtract fractions, using a common denominator.

Example 10 Simplify $\dfrac{3}{x-1}-\dfrac{1}{x+1}$.

Solution Use a common denominator. [You must treat $(x-1)$ and $(x+1)$ as separate expressions with no common factor.]

$$\frac{3}{x-1}-\frac{1}{x+1}=\frac{3(x+1)-(x-1)}{(x-1)(x+1)}$$

$$=\frac{3x+3-x+1}{(x-1)(x+1)}=\frac{2x+4}{(x-1)(x+1)}.$$

Do use brackets, particularly on top – otherwise you are likely to forget the minus at the end of the numerator (in this example subtracting -1 gives +1).

Don't multiply out the brackets on the bottom. You will need to see if there is a factor which cancels out (although there isn't one in this case).

Exercise 1.2

1 Work out the following. Answers may be left as improper fractions.

(a) $\dfrac{4}{7}\times 5$ (b) $\dfrac{5}{12}\times 3$ (c) $\dfrac{7}{9}\times 2$ (d) $\dfrac{4}{15}\times 3$

(e) $\dfrac{8}{11}\div 4$ (f) $\dfrac{8}{11}\div 3$ (g) $\dfrac{6}{7}\div 3$ (h) $\dfrac{6}{7}\div 5$

(i) $\dfrac{3x}{y}\times x$ (j) $\dfrac{3x}{y^2}\times y$ (k) $\dfrac{5x^3}{4y}\div x$ (l) $\dfrac{5x^2}{6y}\div y$

(m) $\dfrac{5x^3}{2y}\times 3x$ (n) $\dfrac{3y^4}{4x^2z}\times 2x$ (o) $\dfrac{6x^2y^3}{5z}\div 2xy$ (p) $\dfrac{5a^2}{6x^3z^2}\div 2y$

2 Make x the subject of the following formulae.

(a) $\frac{1}{2}A=\pi x^2$ (b) $V=\frac{4}{3}\pi x^3$ (c) $\frac{1}{2}(u+v)=tx$ (d) $W=\frac{2}{3}\pi x^2 h$

3 Simplify the following compound fractions.

(a) $\dfrac{\frac{1}{x}+1}{\frac{1}{x}+3}$ (b) $\dfrac{\frac{2}{x}+1}{\frac{3}{x}-1}$ (c) $\dfrac{\frac{1}{x+1}+2}{\frac{1}{x+1}+1}$

4 Write as single fractions.

$\dfrac{2}{x-1}+\dfrac{1}{x+3}$ (b) $\dfrac{2}{x-3}-\dfrac{1}{x+2}$ (c) $\dfrac{1}{2x-1}-\dfrac{1}{3x+2}$ (d) $\dfrac{3}{x+2}+1$

Algebraic fractions answers

1 (a) $\frac{20}{7}$ (b) $\frac{5}{4}$ (c) $\frac{14}{9}$ (d) $\frac{4}{5}$

(e) $\frac{2}{11}$ (f) $\frac{8}{33}$ (g) $\frac{2}{7}$ (h) $\frac{6}{35}$

(i) $\frac{3x^2}{y}$ (j) $\frac{3x}{y}$ (k) $\frac{5x^2}{4y}$ (l) $\frac{5x^2}{6y^2}$

(m) $\frac{15x^4}{2y}$ (n) $\frac{3y^4}{2xz}$ (o) $\frac{3xy^2}{5z}$ (p) $\frac{5a^2}{12x^3yz^2}$

2 (a) $x=\sqrt{\frac{A}{2\pi}}$ (b) $x=\sqrt[3]{\frac{3V}{4\pi}}$ (c) $x=\frac{u+v}{2t}$ (d) $x=\sqrt{\frac{3W}{2\pi h}}$

3 (a) $\frac{1+x}{1+3x}$ (b) $\frac{2+x}{3-x}$ (c) $\frac{3+2x}{2+x}$

4 $\frac{3x+5}{(x-1)(x+3)}$ (b) $\frac{x+7}{(x-3)(x+2)}$ (c) $\frac{x+3}{(2x-1)(3x+2)}$

(d) $\frac{x+5}{x+2}$

Straight line graphs

I am sure that you are very familiar with the equation of a straight line in the form $y = mx + c$, and you have probably practised converting to and from the forms

$$ax + by + k = 0 \qquad \text{or} \qquad ax + by = k,$$

usually with a, b and k are integers. You need to be fluent in moving from one form to the other. The first step is usually to get rid of fractions by multiplying both sides by a common denominator.

Example 1 Write $y = \frac{3}{5}x - 2$ in the form $\boldsymbol{ax + by + k = 0}$, where a, b and k are integers.

Solution Multiply both sides by 5: $5y = 3x - 10$

Subtract $5y$ from both sides: $0 = 3x - 5y - 10$

or $3x - 5y - 10 = 0$

In the first line it is a very common mistake to forget to multiply the 2 by 5.

It is a bit easier to get everything on the right instead of on the left of the equals sign, and this reduces the risk of making sign errors.

In plotting or sketching lines whose equations are written in the form $ax + by = k$, it is useful to use the *cover-up rule*:

Example 2 Draw the graph of $3x + 4y = 24$.

Solution Put your finger over the "$3x$". You see "$4y = 24$".

This means that the line hits the y-axis at $(0, 6)$.

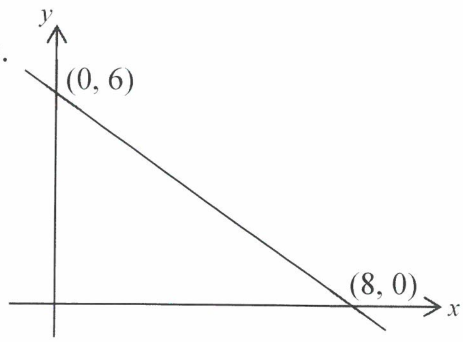
Repeat for the "$4y$". You see "$3x = 24$".

This means that the line hits the x-axis at $(8, 0)$.

[NB: *not* the point $(8, 6)$!]

Mark these points in on the axes.

You can now draw the graph.



Exercise 3.1

1 Rearrange the following in the form $ax + by + c = 0$ or $ax + by = c$ as convenient, where a, b and c are integers and $a > 0$.

(a) $y = 3x - 2$ (b) $y = \frac{1}{2}x + 3$

(c) $y = -\frac{3}{4}x + 3$ (d) $y = \frac{7}{2}x - \frac{5}{4}$

(e) $y = -\frac{2}{3}x + \frac{3}{4}$ (f) $y = \frac{4}{7}x - \frac{2}{3}$

2 Rearrange the following in the form $y = mx + c$. Hence find the gradient and the y-intercept of each line.

(a) $2x + y = 8$ (b) $4x - y + 9 = 0$

(c) $x + 5y = 10$ (d) $x - 3y = 15$

(e) $2x + 3y + 12 = 0$ (f) $5x - 2y = 20$

(g) $3x + 5y = 17$ (h) $7x - 4y + 18 = 0$

3 Sketch the following lines. Show on your sketches the coordinates of the intercepts of each line with the x-axis and with the y-axis.

(a) $2x + y = 8$ (b) $x + 5y = 10$

(c) $2x + 3y = 12$ (d) $3x + 5y = 30$

(e) $3x - 2y = 12$ (f) $4x + 5y + 20 = 0$

Straight line graph answers

1 (a) $3x - y = 2$ (b) $x - 2y + 6 = 0$

(c) $3x + 4y = 12$ (d) $14x - 4y = 5$

(e) $8x + 12y = 9$ (f) $12x - 21y = 14$

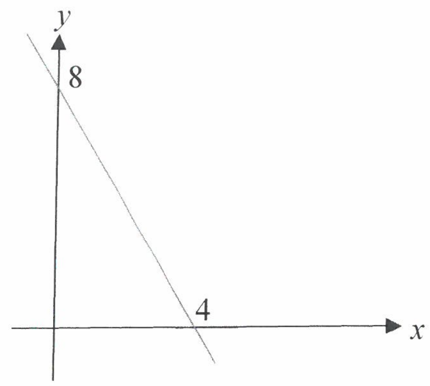
2 (a) $y = -2x + 8$; -2, 8 (b) $y = 4x + 9$; 4, 9

(c) $y = -\frac{1}{5}x + 2$; $-\frac{1}{5}$, 2 (d) $y = \frac{1}{3}x - 5$; $\frac{1}{3}$, -5

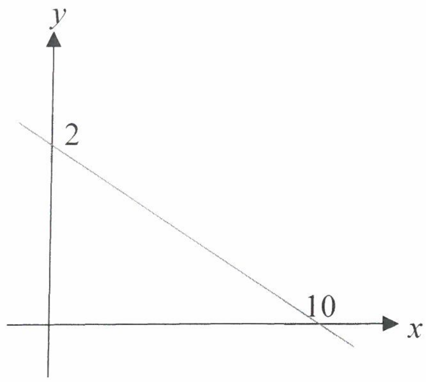
(e) $y = -\frac{2}{3}x - 4$; $-\frac{2}{3}$, -4 (f) $y = \frac{5}{2}x - 10$; $\frac{5}{2}$, -10

(g) $y = -\frac{3}{5}x + \frac{17}{5}$; $-\frac{3}{5}$, $\frac{17}{5}$ (h) $y = \frac{7}{4}x + \frac{9}{2}$; $\frac{7}{4}$, $\frac{9}{2}$

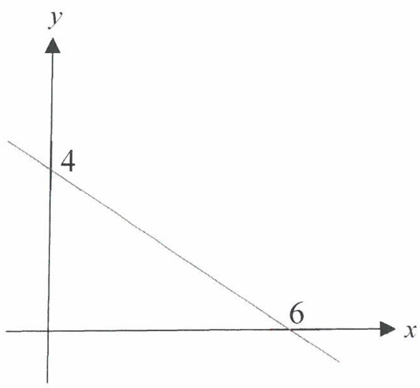
3 (a)



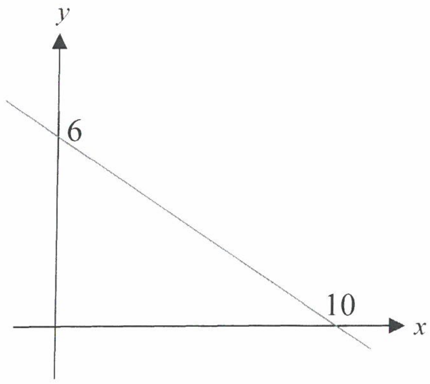
(b)



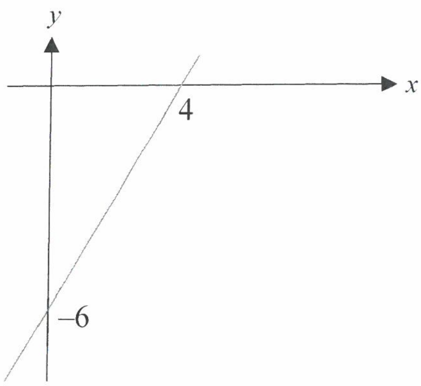
(c)



(d)



(e)



(f)